Perturbative QCD and tau-decays

K. Chetyrkin (with P. A. Baikov and J. H. Kühn)

• $\langle jj^\dagger \rangle$ correlator and $\tau$ decays
• structure of the correlator: massless versus $O(m_q^2)$ contributions
• calculations: status of the art
• $O(\alpha_s^4 N_f^2)$ term in $R_\tau \Rightarrow$ FAC/PMS $\Rightarrow$ contour-improvement

• full $O(\alpha_s^3 m_q^2/s)$ contribution to the correlator and $R_\tau$: results and comparison to the earlier predictions from PMS and FAC and phenomenological applications

• summary

TAU-04
\( \tau \) decays probe the correlator of the charged weak currents in an interesting region of energies just above 1 GeV

\[ \Downarrow \]

strong dependence on \( \alpha_s \) and (for Cabibbo-suppressed part)

on \( m_s \)

\[ \Downarrow \]

good for finding \( \alpha_s \) and \( m_s \)

\[ \Downarrow \]

\( \alpha_s \) is not very small \( \Rightarrow \) higher order QCD terms are important

\( \Rightarrow \) they should be computed and understood
\[ R_\tau = R_{\tau,NS} + R_{\tau,S} \iff \langle jj^\dagger \rangle \text{ correlator} \]

\[ R_\tau \sim 6i\pi \int_{|s|=M_\tau^2} ds \left( 1 - \frac{s}{M_\tau^2} \right)^2 \left[ \Pi^{(q)}(s) - \frac{2}{M_\tau^2} \Pi^{[g]}(s) \right] \]

where

\[ i \int dx e^{iqx} \langle T[j_\mu(x)(j_\nu)^\dagger(0)] \rangle = g_{\mu\nu}\Pi^{(g)}(q^2) + q_\mu q_\nu \Pi^{(q)}(q^2) \]

consider the structure of \( O(m_s^2) \) term (\( Q^2 \equiv -q^2, \; L \equiv \text{Log} \left( \frac{\mu^2}{Q^2} \right) \)):

\[ \Pi^{(g)} = \frac{3}{16\pi^2} \left( Q^2 \Pi^{(g)}(L, \alpha_s) + m_s^2 \Pi_2^{(g)}(L, \alpha_s) + O(m_s^4) \right) \]

\[ \Pi^{(q)} = \frac{3}{16\pi^2} \left( \Pi^{(g)}(L, \alpha_s) + m_s^2 \Pi_2^{(q)}(L, \alpha_s) + O(m_s^4) \right) \]

constant parts of \( \Pi^{(g)} \) and \( \Pi_2^{(g)} \) does not contribute to \( R_{\tau,s} \) while that of \( \Pi_2^{(q)} \) does! \( \longrightarrow \) up to "today" \( R_{\tau,S} \) has been completely known only to order \( a_s^2 \).
consider $m_s = 0$:

$\alpha_s^4$ requires absorptive part of 5-loop correlator

$\equiv$ divergent part $(1/\epsilon)$ of 5-loop correlator

A finite part of 4-loop $\Rightarrow$ div. part of 5-loop

systematic, automatized algorithm /K.Ch. (97) / to express div part of any (L+1)-loop diagram contributing to to a massless correlator in terms of properly constructed set of L-loop massless propagators

B finite part of 3-loop massless propagators: easy $\Rightarrow$ solved more than 20 years ago through integration by parts /K.Ch., Tkachov (81)/

C finite part of 4-loop massless propagators difficult! $\Rightarrow$ not yet completely solved

compare 3- and 4-loop cases
recursion relations based on integration by parts identities!
reduction algorithm and program constructed “manually” for 14 topologies.

4-loop:
much more complicated identities
\sim 150 topologies . . .
straightforward generalization of MINCER
difficult or even impossible!
Baikov: recurrence relation can be solved "mechanically" through $1/D$ expansion\(^1\)

- coefficient functions in front of master integrals depend on $D$ in simple way:

$$C^\alpha(D) = \frac{P^n(D)}{Q^m(D)} \xrightarrow{D\to\infty} \sum_k C^\alpha_k \, (1/D)^k$$

- The terms in the $1/D$ expansion expressible through simple Gaussian integrals (important: a new representation of Feynman amplitudes)

- sufficiently many terms in $1/D$ and $C^\alpha_k \longrightarrow C^\alpha(D)$

Pluses and Minuses of the $1/D$ expansion

+ easy to automatize, simple (relatively) programming

+ (semi)-universality: the idea is applicable to *any* (one-scale?) problem

+ unlike *all* others approaches allows *naturally* to compute directly the sum of all separate t-integrals (within a given topology) \(\Rightarrow\) huge gain in efficiency

+ requires no fancy treatment of polynomials in \(D\) (factorization, etc.) \(\Rightarrow\) a straightforward implementation with FORM3 (including its parallel version)

- hardly be applicable for multiscale problems

- requires a *lot* of computer resources; if CF’s proves to have very complicated \(D\)-dependence might fail due to practical reasons (hardware resources, time, etc.)
status of $R(s)$ at 5 loops

$n_f^2$-terms done: $\Rightarrow$ leading and subleading $n_f$ terms for $R_{ee}$, $R_\tau$, (including $m^2/s$-terms):

$$\alpha_s^4 n_f^3$$

(renormalon chain: purely Abelian simple, long-known)

\[
\begin{array}{c}
\quad + 1 \text{ more}
\end{array}
\]

$$\alpha_s^4 n_f^2$$ (new, non-simple):

\[
\begin{array}{c}
\quad + \sim 100 \text{ more}
\end{array}
\]
RESULTS


\[ R_\tau, \ m = 0 \]

fixed order:

consider \( D_0^{[g]}(Q^2) \equiv -\frac{3}{4} Q^2 \frac{d}{dQ^2} \Pi_0^{[g]} \)

(Adler function, \( \mu \) independent, \( a_s = \alpha_s(Q^2)/\pi \)

\[
D_0^{[g]}(Q^2) = 1 + a_s + a_s^2 \left( -0.1153 n_f + 1.986 \right) \\
+ a_s^3 \left( 0.08621 n_f^2 - 4.216 n_f + 18.24 \right) \\
+ a_s^4 \left( -0.01009 n_f^3 + 1.875 n_f^2 + d_0^{[g]4} n_f + d_0^{[g]4} \right)
\]

\[ 1 \text{Baikov, K.Ch., Kühn, PLR 88 (2002) 012001} \]
use new input: $\alpha_s^4 n_f^2$-term

\[ d_0^{[g]^4}(\text{FAC/PMS}, n_f = 3, 4) = [105.7 - 31.8 n_f] + 1.875 n_f^2 - 0.01009 n_f^3 \]

\[ d_0^{[g]^4}(\text{FAC/PMS}, n_f = 4, 5) = [107.7 - 32.3 n_f] + 1.875 n_f^2 - 0.01009 n_f^3 \]

\[ d_0^{[g]^4}(\text{FAC/PMS}, n_f = 3, 5) = [106.4 - 32.0 n_f] + 1.875 n_f^2 - 0.01009 n_f^3 \]

\[ d_0^{[g]^4}|_{n_f=3} = 27 \pm 16 \quad \text{in full agreement to old prediction by (Kataev, Starshenko)} \]

\[ d_0^{[g]^5}|_{n_f=3} = 145 \pm 100 \]
Implication for $\alpha_s$

with $\alpha_s^4 \rightarrow 0$

\[
\begin{align*}
\alpha_s^{\text{FOPT}}(M_\tau) & = 0.345 \pm (0.025|0.037) \\
\alpha_s^{\text{CIPT}}(M_\tau) & = 0.364 \pm (0.012|0.021) \\
\alpha_s^{\text{FOPT}}(M_Z) & = 0.1209 \pm (0.0024|0.0037) \\
\alpha_s^{\text{CIPT}}(M_Z) & = 0.1229 \pm (0.0011|0.0020)
\end{align*}
\]

with $\alpha_s^4$ and $\alpha_s^5$

<table>
<thead>
<tr>
<th>Method</th>
<th>$\alpha_s(M_\tau)$</th>
<th>$\Delta \delta_P^{\text{exp}}$</th>
<th>$\Delta \mu$</th>
<th>$\Delta d_0^{[g]^4}$</th>
<th>$\Delta d_0^{[g]^5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOPT</td>
<td>$0.330 \pm 0.006 \pm 0.02$</td>
<td>0.006</td>
<td>0.019</td>
<td>0.0045</td>
<td>0.0011</td>
</tr>
<tr>
<td>CIPT</td>
<td>$0.354 \pm 0.009 \pm 0.006$</td>
<td>0.009</td>
<td>0.0036</td>
<td>0.0042</td>
<td>0.0019</td>
</tr>
</tbody>
</table>

Lesson: with any(?) "reasonable" choice of the $\alpha_s^5$ term uncertainty is reduced; difference between FOPT and CIPT remains!

[this difference is reduced for a fictitious heavy lepton of 3 GeV]
\( m_s \) from \( \tau \)-decays

More convenient representation\(^1\) for \( R_{\tau,s} \) \((L + T \equiv (q))\)

\[
R_{\tau} \sim 6i\pi \int_{|s|=M_\tau^2} ds \frac{d}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right) \left[ \left(1 + 2\frac{s}{M_\tau^2}\right)^2 \Pi^{(L+T)}(s) - \frac{2s}{M_\tau^2} \Pi^{(0)}(s) \right]
\]

Facts:

\[ q^2 \Pi^{(0)} \equiv \Pi^g + q^2 \Pi^q \]

\( \Pi^{(0)} = 0 \) in the massless limit; due to a Ward identity it is related to scalar and pseudoscalar correlators

\( \Rightarrow \) and could be constrained from low resonance contributions without a use of pQCD

the PT series for \( L \)-piece is "wilder" than the one for \( L + T \) piece

(\textit{at least for known terms})

\( \Rightarrow \) one could try find \( m_s \) (and or |\( V_{us} \)|) from \( L + T \) contribution only /Maltman, Kambor and Gámiz, Jamin, Pich, Prades, Schwab, . . . /

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\(^1\)Pich, Prades (98)
RESULTS FOR $\Pi_2^{(q)} /2002$ and $2004/$

$$\Pi_2^{(q)} = -4m_s^2 \left( 1 + \frac{7}{3} a_s + a_s^2 \left\{ \left[ -\frac{25}{24} - \frac{2}{9} \zeta_3 \right] n_f + \frac{15331}{432} + \frac{359}{54} \zeta_3 - \frac{520}{27} \zeta_5 \right\} ight)$$

$$+ a_s^3 \left\{ \left[ \frac{2131}{11664} + \frac{19}{81} \zeta_3 \right] n_f^2 + \left[ -\frac{68135}{1944} - \frac{52}{27} \zeta_3^2 - \frac{3997}{486} \zeta_3 - \frac{5}{6} \zeta_4 + \frac{3875}{243} \zeta_5 \right] n_f ight\}$$

$$+ \frac{2629301}{5184} + \frac{293333}{648} \zeta_3 + \frac{653}{18} \zeta_3^2 - \frac{138695}{324} \zeta_5 + \frac{79835}{648} \zeta_7 \right\}$$

$$= -4m_s^2 \left( 1 + 2.333 a_s + a_s^2 \left\{ -1.309 n_f + 23.51 \right\} ight)$$

$$+ a_s^3 \left\{ 0.4647 n_f^2 - 32.08 n_f + (k_{2,0}^{(q)^3} = 294.38) \right\}$$

$$= -4m_s^2 (1. + 2.33333 a_s + 19.583 a_s^2 + 202.309 a_s^3)$$

the very calculation took (very roughly!) about 2 PC-years!
Comparison to PMS/FAC/NNA predictions

\[ k^{(q),3}(\text{EXACT}) = -202.309 \]

\[ k^{(q),3}(\text{predicted}) = 200(\text{PMS}) \quad 199(\text{FAC}) \quad 127(\text{NNA}) \]

The astonishingly good agreement gives us a strong argument to repeat the game and predict, starting from now completely known \( k^{(q),3}_2 \) the corresponding result for one loop more, that is for \( k^{(q),4}_2 \). To be definite, we cite the PMS predictions (FAC results are very similar)

\[ k^{(q),4}_2 = 2200 \pm 200 \]

\[ ^2 \text{fine print: It is, of course, difficult to assign a qualitative estimate of possible uncertainty in the above predictions; however, a simple comparison to } \alpha^3_s \text{ case strongly suggests that an error of about 10\% should be considered as a conservative one} \]

\[ ^1 \text{P.A.Baikov, K.Ch., J. H. Kühn, PLB 559:245-251,2003} \]
**subtlety: to use** $D_2^{L+T}$ **or** $\Pi_2^{L+T}$ **?**

\[
\Delta^{L+T} = \oint ds P(s) \Pi_2^{L+T}(s) / s \equiv \oint ds \bar{P}(s) (D_2^{L+T}(s) \equiv Q^2 \frac{d}{ds} \Pi_2^{L+T} / s)
\]

\[
/ K. Ch., Kühn, Pivovarov (98)/

\[
\Pi^{L+T} = \Pi_0^{L+T} + \frac{m_s^2}{Q^2} \Pi_2^{L+T} \Leftarrow \text{no subtraction constants for} \ \Pi_2^q \text{is necessary!}
\]

**problem:** RG-improvement **does not commute to** $\frac{d}{ds}$ !

\[
\Pi : \frac{1}{s} \left[ \alpha_s(\mu) + \beta_0 \log \frac{s}{\mu^2} \alpha_s^2(\mu) \right] \xrightarrow{\text{RG-imp}} \alpha_s(s) / s
\]

\[
D : \frac{1}{s} \left[ \alpha_s(\mu) + \beta_0 \log \frac{s}{\mu^2} \alpha_s^2(\mu) \right] \xrightarrow{s \frac{d}{ds} + \text{RG-imp}} -\alpha_s(s) / s - \beta_0 \alpha_s^2(s) / s
\]

**conclusion:** $s \frac{d}{ds}$ moves part of lower order input to **higher orders** $\Rightarrow$ **contrary** to the spirit of CIPT!
this is confirmed by inspecting the convergence pattern:

\[
\Delta^{L+T}(\alpha_s = .15, D_2^{L+T}) = 0.952 + 0.182\ h + 0.0664\ h^2 + 0.0278\ h^3 + 0.0249\ h^4 \\
= 0.952, 1.134, 1.2004, 1.2282, 1.253
\]

\[
\Delta^{L+T}(\alpha_s = .15, \text{direct}) = 1.05 + 0.118\ h + 0.0453\ h^2 + 0.0201\ h^3 + 0.0174\ h^4 \\
= 1.05, 1.168, 1.2133, 1.2334, 1.251
\]

\[
\Delta_{30}^{L+T}(\alpha_s = .334, D_2^{L+T}) = 1.19 + 0.571\ h + 0.48\ h^2 + 0.416\ h^3 + 0.625\ h^4 \\
= 1.19, 1.761, 2.241, 2.657, 3.282
\]

\[
\Delta_{30}^{L+T}(\alpha_s = .334, \text{direct}) = 1.59 + 0.471\ h + 0.413\ h^2 + 0.339\ h^3 + 0.269\ h^4 \\
= 1.59, 2.061, 2.474, 2.813, 3.082
\]

where 4-loop terms come from PMS estimations for \( \Pi_2^{L+T} \)
BUT! LIFE IS NOT SO SIMPLE!!!

\[ \Delta_{20}^{L+T}(\alpha_s = .334, D_{20}^{L+T}) = 1.05 + 0.451 \, h + 0.327 \, h^2 + 0.223 \, h^3 + 0.152 \, h^4 \]
\[ = 1.05, 1.501, 1.828, 2.05, 2.203 \]

\[ \Delta_{20}^{L+T}(\alpha_s = .334, \text{direct}) = 1.35 + 0.347 \, h + 0.247 \, h^2 + 0.12 \, h^3 - 0.223 \, h^4 \]
\[ = 1.35, 1.697, 1.944, 2.064, 1.841 \]

one observes rather significant contribution from \( \mathcal{O}(\alpha_s^4) \) term: it looks like both (or one of) 2 PT series begin to behave itself wildly at this order! (Kambor, Maltman (2000))
<table>
<thead>
<tr>
<th>Parameter</th>
<th>(2,0)</th>
<th>(3,0)</th>
<th>(4,0)</th>
<th>w. aver</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>+33.6</td>
<td>+25.0</td>
<td>+21.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-44.3</td>
<td>-29.5</td>
<td>-23.0</td>
<td></td>
</tr>
<tr>
<td>$m_s(\mathcal{O}(a_s^3), \text{exact})$</td>
<td>92.5</td>
<td>85.3</td>
<td>78.1</td>
<td>82.5 ± 17</td>
</tr>
<tr>
<td>$\mathcal{O}(a_s^3)$</td>
<td>-4.6</td>
<td>-6.0</td>
<td>-6.7</td>
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</tr>
<tr>
<td></td>
<td>+5.5</td>
<td>+7.6</td>
<td>+8.0</td>
<td></td>
</tr>
<tr>
<td>others</td>
<td>+34</td>
<td>+25</td>
<td>+20</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-44</td>
<td>-29</td>
<td>-22</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>+33.6</td>
<td>+25.0</td>
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</tr>
<tr>
<td></td>
<td>-44.3</td>
<td>-29.5</td>
<td>-23.0</td>
<td></td>
</tr>
<tr>
<td>$m_s(\mathcal{O}(a_s^4), \text{PMS})$</td>
<td>89.3</td>
<td>76.8</td>
<td>66.5</td>
<td>73.2 ± 17</td>
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<tr>
<td>$\mathcal{O}(a_s^4)$</td>
<td>-3.0</td>
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<td>-7.6</td>
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<tr>
<td></td>
<td>+3.2</td>
<td>+8.6</td>
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<td>others</td>
<td>+34</td>
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<td></td>
<td>-44.3</td>
<td>-29.8</td>
<td>-23.4</td>
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</tbody>
</table>

Table 1: An update of the Table 1 of [1] /Gámiz,Jamin,Pic,Prades,Schwab (2004)/ for $m_s$ extracted from recent exp.data of the OPAL collaboration /G.Abbiendi et al, (2004)/ with subtracted longitudinal contribution accoding to [1]. The contour improvement has been done with the Adler function $D_{2}^{L+T}$. “others” ⇒ all uncertainties (added in quadrature) of the input parameters different from the $\mathcal{O}(a_s^3)$ (or $\mathcal{O}(a_s^4)$) terms in the perturbative contribution. The last colomun shows a weighted average over the different moments ( as the individual error for a given moment we have chosen the larger one)
<table>
<thead>
<tr>
<th>Parameter</th>
<th>(2,0)</th>
<th>(3,0)</th>
<th>(4,0)</th>
<th>w. aver</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_s(O(a_s^3), \text{exact})$</td>
<td>92.4</td>
<td>83.0</td>
<td>74.2</td>
<td>79.6 ± 17</td>
</tr>
<tr>
<td>$O(a_s^3)$</td>
<td>−2.6</td>
<td>−4.6</td>
<td>−5.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+2.8</td>
<td>+5.5</td>
<td>+7.3</td>
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<tr>
<td>others</td>
<td>+34</td>
<td>+24</td>
<td>+20</td>
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<tr>
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<td>−44</td>
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<td>−22</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>+33.6</td>
<td>+25.0</td>
<td>+21.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>−44.3</td>
<td>−29.5</td>
<td>−23.0</td>
<td></td>
</tr>
<tr>
<td>$m_s(O(a_s^4), \text{PMS})$</td>
<td>97.9</td>
<td>79.3</td>
<td>65.7</td>
<td>74.7 ± 17</td>
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<tr>
<td>$O(a_s^4)$</td>
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<tr>
<td></td>
<td>+6.5</td>
<td>+3.7</td>
<td>+8.4</td>
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<td>others</td>
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<td>+24</td>
<td>+20</td>
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<td></td>
<td>−44</td>
<td>−29</td>
<td>−22</td>
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<tr>
<td>Total</td>
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<tr>
<td></td>
<td>−45</td>
<td>−29</td>
<td>−23</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: The same as Table I but with the contour improvement done directly for $\Pi^L+T$

$\Rightarrow$ at $O(a_s^4)$ weighted averages of both tables are close and lead to

$$m_s(M_\tau)_{5\text{-loops}} = 74 \pm 23 \text{ MeV}$$

which should be compared to

$$m_s(M_\tau)_{4\text{-loops}} = 84 \pm 23 \text{ MeV} \text{ according to /Gámiz, et al (2004)/}$$
Summary: $\alpha_s$

- $\alpha_s^4$-terms for $R_{e^+e^-}$ and $R_{\tau}$ are important for improved determination of $\alpha_s$

- subleading $n_f$ terms are available

- reasonable agreement with previous estimates

  $\Rightarrow$ improved value for $\alpha_s$

- complete calculation of $\alpha_s^4$-terms for $R_{\tau}$ and $R_{e^+e^-}$ in the massless limit is possible and is currently under way

- difference between CIPT and FOPT results seems to persist in higher orders
Summary: $m_s$

- analytical QCD result for at $m_s^2 \alpha_s^3$ order contribution to $R_\tau$ is available

- comparison of the exact result to predictions from various "optimization schemes" demonstrate striking success (relative accuracy around 1(!) percent) of PMS and FAC

- the success strongly suggests to consider and to use the PMS prediction for $m_s^2 \alpha_s^4$ as quite reliable one

- pure convergence of the PT series for $m_s \Rightarrow$ requires new ideas (clever than $L + T$ choice of the integration weight? /Kambor, Maltman (2000)/)

- accurate measurements of lower moments of $R_{\tau,S}$ are important to decrease unphysical dependence of $m_s$ from the moment

- no way to compute $m_s^2 \alpha_s^4$-term in any foreseeable future