Muon $g-2$: a theoretical review

Tau04
Nara, September 2004

Andrzej Czarnecki 🍁 University of Alberta
Outline

QED: present and future (T. Kinoshita)
Electroweak loops
Hadronic effects
  * vacuum polarization
    (J. Kühn, S. Eidelman, D. Leone, M. Davier, B. Schwarz, K. Hagiwara)
  * light-by-light scattering
Summary and outlook
**Muon $g-2$: Standard Model update**

*Units: $10^{-11}$*

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>QED</strong></td>
<td>116 584 719 (1)</td>
<td>hep-ph/0402206</td>
</tr>
<tr>
<td><strong>Hadronic</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LO</td>
<td>6 963 (72)</td>
<td>hep-ph/0308213</td>
</tr>
<tr>
<td>NLO</td>
<td>- 98 (1)</td>
<td>hep-ph/0312250</td>
</tr>
<tr>
<td>LBL</td>
<td>120 (40)</td>
<td>tentative, see hep-ph/0312226</td>
</tr>
<tr>
<td><strong>Electroweak</strong></td>
<td>154 (3)</td>
<td>hep-ph/0212229</td>
</tr>
<tr>
<td><strong>Total SM</strong></td>
<td>116 591 858 (82)</td>
<td></td>
</tr>
</tbody>
</table>

**Experiment - SM Theory = 222 (102) (2.2σ deviation)**
Muon $g-2$: new data

Brookhaven, January 2004: $\mu$ measurement.

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 222 \pm 102 \cdot 10^{-11}$$
$$\rightarrow 2.2\sigma$$
(based on $e^+e^-$)

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 123 \pm 89 \cdot 10^{-11}$$
$$\rightarrow 1.4\sigma$$
(tau)

from A. Vainshtein
QED contributions: muon vs. electron

Enhancement factors:

\[ \pi^{2n} \ln \frac{m_\mu}{m_e} \]

\[ \ln^n \frac{m_\mu}{m_e} \]

Leading five-loop effects must be included!
QED contributions: problems at 4-loop order

Traditional approach (T. Kinoshita and M. Nio (Nara)): numerical problems, digit deficiency

New approach (various groups, in progress):
Combine numerical and algebraic methods
Reduce all integrals to a smaller basis
Evaluate the primitive integrals numerically, with high accuracy.
Example: integration by parts

\[ J(a_1, a_2) = \int d^Dk \frac{1}{(k^2)^{a_1} (k^2 + 2kp)^{a_2}} \]

\[ 0 = p_\mu \int d^Dk \frac{\partial}{\partial k_\mu} \frac{1}{(k^2)^{a_1} (k^2 + 2kp)^{a_2}} \]
New approach to the QED part

Obstacles:

- Very large number of integrals
- Reduction to primitive integrals
- Evaluation of master integrals

Recent progress: algorithmic reduction (Laporta, 2001)
Electroweak effects: pure and hadronic

Small part of the total $g-2$: $154(3) \times 10^{-11}$

\[ ( -1 ) \cdot \left( \frac{5 G_{\mu} m^2}{24 \sqrt{2} \pi^2} \right) + 2 \cdot \frac{5 G_{\mu} m^2}{24 \sqrt{2} \pi^2} \approx 195 \cdot 10^{-11} \]
Higher-order electroweak effects

Most important: photonic corrections → large logs

\[
\frac{\alpha}{\pi} G_\mu m_\mu^2 \ln \frac{M_W^2}{m_\mu^2} \sim -23\% \text{ of one-loop }
\]

Kukhto et al.
AC, Krause, Marciano
Heinemaier, Stockinger, Weiglein
Muon $g-2$: hadronic loops

Hadronic effects dominate theoretical uncertainty:
- Vacuum polarization
- Light-by-light scattering
- Electroweak triangle diagrams (numerically small)
Electroweak-hadronic effects

Large logs $\ln(m_\mu/M_Z)$ appear in individual fermion contributions;

But cancel in the sum for each generation - like anomalies.

This cancellation between leptons and hardons was controversial.

AC, Marciano, Vainshtein
vs.
Knecht, Peris, Perrottet, de Rafael

Useful illustration: similar techniques used in light-by-light
Structure of the triangle

\[ \sim w_T(q^2)(-q^2 \tilde{F}_{\mu\nu} + q_\mu q_\sigma \tilde{F}_{\sigma\nu} - q_\nu q_\sigma \tilde{F}_{\sigma\mu}) + w_L(q^2)(q_\nu q_\sigma \tilde{F}_{\sigma\mu}) \]

Perturbative result:

\[ w_L = 2w_T \sim \frac{1}{q^2} \]

Anomaly:

\[ q_\nu T^{\mu\nu} \sim q^2 w_L \neq 0 \]

\[ q_\mu T^{\mu\nu} = 0 \]
Vainshtein’s non-renormalization theorem for \( w_T \)

\[
T_{\mu\nu} \sim w_T \left( q^2 \right) \left( -q^2 \tilde{F}_{\mu\nu} + q_\mu q^\sigma \tilde{F}_{\sigma\nu} - q_\nu q^\sigma \tilde{F}_{\sigma\mu} \right) + w_L \left( q^2 \right) \left( q_\nu q^\sigma \tilde{F}_{\sigma\mu} \right)
\]

In the chiral limit, \( w_T \) has no perturbative corrections.

Idea of the proof:

\[
\text{Im} T_{\mu\nu} \sim q_\mu q^\sigma \tilde{F}_{\sigma\nu} + q_\nu q^\sigma \tilde{F}_{\sigma\mu} \quad \text{(symmetric)}
\]

\[
2w_T \left( q^2 \right) = w_L \left( q^2 \right)
\]

Guidance from one-loop calculations!
\( w_{T,L} \) in QCD (chiral limit)

Perturbative:
\[
W_L = 2W_T = \frac{2}{Q^2} \quad Q^2 \equiv -q^2
\]

Non-perturbative:

<table>
<thead>
<tr>
<th>Large ( Q^2 )</th>
<th>Small ( Q^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_L )</td>
<td>( w_T )</td>
</tr>
<tr>
<td>( \frac{2}{Q^2} )</td>
<td>( \frac{2}{Q^2} )</td>
</tr>
<tr>
<td>( \frac{1}{Q^2} - \frac{\left(0.7 \text{ GeV}\right)^4}{Q^6} + O\left(\frac{1}{Q^8}\right) )</td>
<td>( \frac{1}{m_{a_1}^2 - m_\rho^2} \left( \frac{m_{a_1}^2 - m_\pi^2}{Q^2 + m_\rho^2} - \frac{m_\rho^2 - m_\pi^2}{Q^2 + m_{a_1}^2} \right) )</td>
</tr>
</tbody>
</table>

(pion pole)

(model for \( w_T \))
Contributions to $g-2$

$$\Delta a_\mu \sim \left( \frac{\alpha}{\pi} \right)^2 \frac{m_\mu^2}{M_Z^2} \int dQ^2 \left( w_L + \frac{M_Z^2}{M_Z^2 + Q^2} w_T \right)$$

Asymptotics:

$$w_{T,L} \xrightarrow{Q^2 \to \infty} \frac{1}{Q^2} \sum_f I_{3f} N_f Q_f^2$$

$$\int dQ^2 w_L : \text{diverges} \quad \text{theory inconsistent unless anomalies cancel}$$

$$\int dQ^2 \frac{M_Z^2}{M_Z^2 + Q^2} w_T \sim \ln M_Z^2$$
"Pure" hadronic contributions

Recent progress

Updated studies of g-2 using $e^+e^-$ data

Davier, Eidelman, Höcker, Zhang
Hagiwara, Martin, Nomura, Teubner

Novosibirsk results tested by Daphne

See talks by Kühn, Leone, Shwartz

Shift of the light-by-light prediction

Melnikov and Vainshtein
Vacuum polarization: $\tau$ decays vs. $e^+e^-$

$W$: $l=1$ & $V, A$

$CVC$: $l=1$ & $V$

$\gamma$: $l=0, 1$ & $V$

From M. Davier, A. Hoecker
Vacuum polarization: $e^+e^-$

$e^+e^-$ data have greatly improved
New results from KLOE confirm Novosibirsk CMD2
Hadronic contributions: outstanding problems

How to reconcile $e^+e^-$ and $\tau$ data?

Can we improve the light-by-light light prediction?
Light-by-light scattering

Recent evaluations:

Knecht, Nyffeler 80(40)
Hayakawa, Kinoshita 90(15)
Bijnens, Pallante, Prades 83(32)
Melnikov, Vainshtein 136(25)
Effects enhanced by $N_c$

Quark box: pQCD asymptotics

The same structure as in the EW-hadronic loops.

Dominant contribution: $\pi^0$ pole
Crucial observation: $1/Q^2$ asymptotics $\rightarrow$ no formfactor in $\pi^*\gamma^*\gamma$ if one of the photons soft.

$$\Delta a^\text{PS}_\mu \simeq (76.5 + 2\cdot18) \cdot 10^{-11}$$

$$\Delta a^\text{PV}_\mu \simeq 22 \cdot 10^{-11}$$
What about terms subleading in $N_c$?

Example: pion box. It is chirally enhanced, $\frac{m_\mu^2}{m_\pi^2}$

Numerical effect: small.
Previous estimates:
- $-4.5(8.5)\times10^{-11}$ HLS Hayakawa, Kinoshita, Sanda
- $-19(5)\times10^{-11}$ VMD Bijnens, Pallante, Prades

Melnikov & Vainshtein: $0\pm10\times10^{-11}$
(cancellations with higher orders in the chiral expansion)
Summary on the light-by-light scattering

Matching of hadronic model with perturbative QCD, at asymptotic momentum transfer. Large contribution of high virtualities.

Dominant in $N_c \to \infty$: pion pole

Still room for improvement: subleading terms
Pomeranchuk and Sakharov on $g - 2 = \frac{\alpha}{\pi}$

If this is true, it’s exceptionally important; if it isn’t true, that, too, is exceptionally important. (Pomeranchuk after Sakharov’s talk, 1949)

I felt like the messenger of the gods. (Sakharov)
How do we determine $g-2$?

Measure $\omega_a = \frac{g-2}{2} \frac{e}{m_\mu} B$

$B$ from NMR: $\omega_p = \frac{2 \mu_p B}{\hbar}$

$\frac{e}{m_\mu}$ from $\mu_\mu = g \frac{e\hbar}{4m_\mu}$

Master formula: $\frac{g-2}{2} = \frac{\omega_a / \omega_p}{\mu_\mu / \mu_p - \omega_a / \omega_p}$

Measured by E821

From muonium
Muonium spectrum determines $\mu_\mu/\mu_p$

$\nu_{34} - \nu_{12} \sim \mu_\mu B \quad \Rightarrow \quad \mu_\mu/\mu_p$

Measured to relative $1.2 \cdot 10^{-7}$ (like $15 \cdot 10^{-11}$ in $a_\mu$)

Will need improvement for the "next g-2"

Mu: also $m_\mu/m_e$ and tests of QED