Perturbative QCD and tau-decays

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- $\langle jj^\dagger \rangle$ **correlator and** τ **decays**
- **structure of the correlator: massless versus** $\mathcal{O}(m_q^2)$ **contributions**
- **calculations: status of the art**
- $O(\alpha_s^4 N_f^2)$ **term in** $R_\tau \Rightarrow$ **FAC/PMS** $\Rightarrow$ **contour-improvement**

- **full** $\mathcal{O}({\alpha_s}^3 m_q^2/s)$ **contribution to the correlator and** R_τ**: results and comparison to the earlier predictions from PMS and FAC and phenomenological applications**

- **summary**

TAU-04

τ decays probe the correlator of the charged weak currents in an interesting region of energies just above 1 GeV

$\Downarrow$

strong dependence on α_s and (for Cabibbo-suppressed part) on m_s

$\Downarrow$

good for finding α_s and m_s

$\Downarrow$

α_s is not very small $\Rightarrow$ higher order QCD terms are important $\Rightarrow$ they should be computed and understood

$R_\tau = R_{\tau,NS} + R_{\tau,S} \Longleftrightarrow \langle j j^\dagger \rangle$ correlator

$$R_\tau \sim 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2 \left[\Pi^{(q)}(s) - \frac{2}{M_\tau^2}\Pi^{[g]}(s)\right]$$

where

$$i\int dx\, e^{iqx}\langle T[\, j_\mu(x)(j_\nu)^\dagger(0)\,]\rangle = g_{\mu\nu}\Pi^{(g)}(q^2) + q_\mu q_\nu \Pi^{(q)}(q^2)$$

consider the structure of $\mathcal{O}(m_s^2)$ term ($Q^2 \equiv -q^2,\ L \equiv Log\ \left(\frac{\mu^2}{Q^2}\right)$):

$$\Pi^{(g)} = \ \frac{3}{16\pi^2}\ (Q^2\Pi^{(g)}(L,\alpha_s) + m_s^2\ \Pi_2^{(g)}(L,\alpha_s) + \mathcal{O}(m_s^4)$$

$$\Pi^{(q)} = \ \frac{3}{16\pi^2}\ (\Pi^{(g)}(L,\alpha_s) + m_s^2\ \Pi_2^{(q)}(L,\alpha_s) + \mathcal{O}(m_s^4)$$

constant parts of $\Pi^{(g)}$ and $\Pi_2^{(g)}$ does not contribute to $R_{\tau,s}$ while that of $\Pi_2^{(q)}$ does! $\longrightarrow$ up to "today" $R_{\tau,S}$ has been completely known only to order a_s^2

consider $m_s = 0$:

α_s^4 requires absorptive part of 5-loop correlator

$\widehat{=}$ divergent part $(1/\epsilon)$ of 5-loop correlator

A finite part of 4-loop $\Rightarrow$ div. part of 5-loop

systematic, automatized algorithm /K.Ch. (97) / to express div part of any (L+1)-loop diagram contributing to to a massless correlator in terms of properly constructed set of L-loop massless propagators

B finite part of 3-loop massless propagators: easy $\Rightarrow$ solved more than 20 years ago through integration by parts /K.Ch., Tkachov (81)/

C finite part of 4-loop massless propagators difficult! $\Rightarrow$ not yet completely solved

compare 3- and 4-loop cases

MINCER: 3-loop /Larin, Tkachov, Vermaseren (92)/

recursion relations based on integration by parts identities!

reduction algorithm and program constructed "manually" for 14 topologies.

4-loop:

much more **complicated identities**

$\sim$ **150 topologies . . .**

straightforward generalization of MINCER

difficult or even impossible!

Baikov: reccurence relation can be solved "mechanically" through $1/D$ expansion[1]

- coefficient functions in front of *master integrals* depend on D in simple way:

$$C^{\alpha}(D) = \frac{P^n(D)}{Q^m(D)} \underset{D\to\infty}{=\!=\!=} \sum_k C^{\alpha}_k \ (1/D)^k$$

- The terms in the $1/D$ expansion expressible through simple Gaussian integrals (important: a new representation of Feynman amplitudes)

- sufficiently many terms in $1/D$ and $C^{\alpha}_k \longrightarrow C^{\alpha}(D)$

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[1] Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

Pluses and Minuses of the $1/D$ expansion

+ easy to automatize, simple (relatively) programming

+ (semi)-universality: the idea is applicable to *any* (one-scale?) problem

+ unlike *all* others approaches allows *naturally* to compute directly the sum of all separate t-integrals (within a gi ven topology) $\Longrightarrow$huge gain in efficiency

+ requires no fancy treatment of polynomials in D (factorization, etc.) $\Longrightarrow$ a straightforward implementation with FORM3 (including its parallel version)

- hardly be applicable for multiscale problems

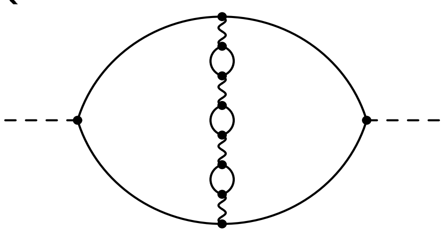
- requires **a lot** of computer resources; if CF's proves to have very complicated D-dependence might fail due to practical reasons (hardware resources, time, etc.)

status of $R(s)$ at 5 loops

n_f^2-terms done: $\Rightarrow$ leading and subleading n_f terms for $R_{e^+e^-}$, R_τ, (including m^2/s-terms):

$\alpha_s^4 n_f^3$

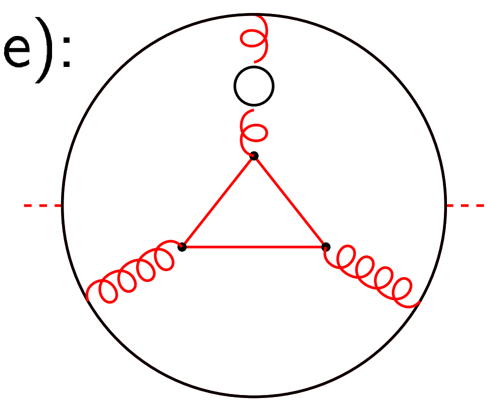
(renormalon chain: purely Abelian simple, long-known)

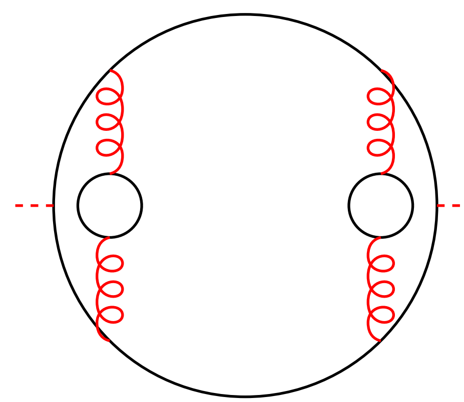


$+$ 1 more

$+$

$\alpha_s^4 n_f^2$ (new, non-simple):





$+ \sim$ 100 more

RESULTS[1]

R_τ, $m = 0$

fixed order:

consider $D_0^{[g]}(Q^2) \equiv -\frac{3}{4} Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} \Pi_0^{[g]}$

(Adler function, μ **independent,** $a_s = \alpha_s(Q^2)/\pi$

$$
\begin{aligned}
D_0^{[g]}(Q^2) = 1 + \; & a_s + a_s^2 \left(-0.1153\, n_f + 1.986\right) \\
& + a_s^3 \left(0.08621\, n_f^2 - 4.216\, n_f + 18.24\right) \\
& + a_s^4 \left(-0.01009\, n_f^3 + \mathbf{1.875}\, n_f^2 + d_{0,1}^{[g]4}\, n_f + d_{0,0}^{[g]4}\right)
\end{aligned}
$$

[1] Baikov, K.Ch., Kühn, PLR 88 (2002) 012001

use new input: $\alpha_s^4 n_f^2$-term

$$d_0^{[g]4}(\mathrm{FAC/PMS}, n_f = 3, 4) = \boxed{105.7 - 31.8\, n_f} + 1.875\, n_f^2 - 0.01009\, n_f^3$$

$$d_0^{[g]4}(\mathrm{FAC/PMS}, n_f = 4, 5) = \boxed{107.7 - 32.3\, n_f} + 1.875\, n_f^2 - 0.01009\, n_f^3$$

$$d_0^{[g]4}(\mathrm{FAC/PMS}, n_f = 3, 5) = \boxed{106.4 - 32.0\, n_f} + 1.875\, n_f^2 - 0.01009\, n_f^3$$

$\Downarrow$ $\Downarrow$

$d_0^{[g]4}|_{n_f=3} = 27 \pm 16$ **in full agreement to old prediction by (Kataev, Starshenko)**

$$d_0^{[g]5}|_{n_f=3} = 145 \pm 100$$

Implication for α_s

with $\alpha_s^4 \to 0$

$$\begin{aligned} \alpha_s^{\rm FOPT}(M_\tau) &= 0.345 \pm (0.025|0.037) \\ \alpha_s^{\rm CIPT}(M_\tau) &= 0.364 \pm (0.012|0.021) \\ \alpha_s^{\rm FOPT}(M_Z) &= 0.1209 \pm (0.0024|0.0037) \\ \alpha_s^{\rm CIPT}(M_Z) &= 0.1229 \pm (0.0011|0.0020) \end{aligned}$$

with α_s^4 and α_s^5

Method	$\alpha_s(M_\tau)$	$\Delta\,\delta_P^{\rm exp}$	$\Delta\,\mu$	$\Delta\,d_0^{[g]4}$	$\Delta\,d_0^{[g]5}$
FOPT	$0.330 \pm 0.006 \pm 0.02$	0.006	0.019	0.0045	0.0011
CIPT	$0.354 \pm 0.009 \pm 0.006$	0.009	0.0036	0.0042	0.0019

Lesson: with any(?) "reasonable" choice of the ${\alpha_s}^5$ term uncertainty is reduced; difference between FOPT and CIPT remains!

[this difference is reduced for a fictitious heavy lepton of 3 GeV]

$\boldsymbol{m_s}$ from τ-decays

More convenient representation[1] for $R_{\tau,S}$ $\quad (L+T \equiv (q))$

$$R_\tau \sim 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)\left[\left(1+2\frac{s}{M_\tau^2}\right)^2 \Pi^{(L+T)}(s) - \frac{2s}{M_\tau^2}\Pi^{(0)}(s)\right]$$

Facts:

$$q^2\Pi^{(0)} \equiv \Pi^g + q^2\Pi^q$$

$\Pi^{(0)} = 0$ in the massless limit; due to a Ward identity it is related to scalar and pseudoscalar correlators

$\Rightarrow$ and could be constrained from low resonance contributions without a use of pQCD

the PT series for L-piece is "wilder" than the one for $L+T$ piece
(**at least for known terms)**

$\Rightarrow$ one could try find m_s (and or $|V_{us}|$) from $L+T$ contribution only /Maltman, Kambor and Gámiz,Jamin, Pich, Prades, Schwab, . . . /

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[1] Pich, Prades (98)

RESULTS FOR $\Pi_2^{(q)}$ /2002 and 2004/

$$\Pi_2^{(q)} = -4m_s^2 \left(1 + \frac{7}{3} a_s + a_s^2 \left\{ \left[-\frac{25}{24} - \frac{2}{9}\zeta_3 \right] n_f + \frac{15331}{432} + \frac{359}{54}\zeta_3 - \frac{520}{27}\zeta_5 \right\} \right.$$

$$+ a_s^3 \left\{ \left[\frac{2131}{11664} + \frac{19}{81}\zeta_3 \right] n_f^2 + \left[-\frac{68135}{1944} - \frac{52}{27}\zeta_3^2 - \frac{3997}{486}\zeta_3 - \frac{5}{6}\zeta_4 + \frac{3875}{243}\zeta_5 \right] n_f \right.$$

$$\left. + \frac{2629301}{5184} + \frac{29333}{648}\zeta_3 + \frac{653}{18}\zeta_3^2 - \frac{138695}{324}\zeta_5 + \frac{79835}{648}\zeta_7 \right\}$$

$$= -4m_s^2 \left(1 + 2.333\, a_s + a_s^2 \{ -1.309\, n_f + 23.51 \} \right.$$

$$\left. + a_s^3 \left\{ 0.4647\, n_f^2 - 32.08\, n_f + (k_{2,0}^{(q)3} = 294.38) \right\} \right)$$

$$= -4m_s^2 (1. + 2.33333\, a_s + 19.583\, a_s^2 + 202.309\, a_s^3)$$

the very calculation took (very roughly!) about 2 PC-years!

Comparison to PMS/FAC/NNA predictions[1]

$$\boldsymbol{k^{(q),3}}(\mathrm{EXACT}) = -\mathbf{202.309}$$

$$\boldsymbol{k^{(q),3}(predicted)} = \quad \mathbf{200}(\mathrm{PMS}) \quad \mathbf{199}(\mathrm{FAC}) \quad \mathbf{127}(\mathrm{NNA})$$

The astonishingly good agreement gives us a strong argument to repeat the game and predict, starting from now completely known $k_2^{(q)3}$ the corresponding result for one loop more, that is for $k_2^{(q)4}$. To be definite, we cite the PMS predictions (FAC results are very similar)

$$k_2^{(q)4} = 2200 \pm 200^2$$

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[2] fine print: It is, of course, difficult to assign a qualitative estimate of possible uncertainty in the above predictions; however, a simple comparison to α_s^3 case strongly suggests that an error of about 10% should be considered as a conservative one

[1]P.A.Baikov, K.Ch., J. H. Kühn, PLB 559:245-251,2003

subtlety: to use D_2^{L+T} or Π_2^{L+T} ?

$$\Delta^{L+T} = \oint ds P(s) \Pi_2^{L+T}(s)/s \equiv \oint ds \bar{P}(s) (D_2^{L+T}(s) \equiv Q^2 \frac{d}{ds} \Pi_2^{L+T}/s)$$

$\Downarrow$ $\Downarrow$

/K.Ch.,Kühn, Pivovarov (98)/ /Pich, Prades (98)/

$\Pi^{L+T} = \Pi_0^{L+T} + \frac{m_s^2}{Q^2} \Pi_2^{L+T} \Leftarrow$ no subtraction constants for Π_2^q is necessary!

problem: RG-improvement does not commute to $\frac{d}{ds}$!

$$\Pi : \frac{1}{s}\left[\alpha_s(\mu) + \beta_0 \, Log \frac{s}{\mu^2} \, {\alpha_s}^2(\mu)\right] \xrightarrow{\mathrm{RG-imp}} \alpha_s(s)/s$$

$$\boldsymbol{D : \frac{1}{s}\left[\alpha_s(\mu) + \beta_0 \, Log \frac{s}{\mu^2} \, {\alpha_s}^2(\mu)\right] \xrightarrow{\mathrm{s\frac{d}{ds}+RG-imp}} -\alpha_s(s)/s - \beta_0 {\alpha_s}^2(s)/s}$$

conclusion: $s\frac{d}{ds}$ moves part of lower order input

to higher orders $\Rightarrow$ contrary to the spirit of

CIPT!

this is confirmed by inspecting the convergence pattern:

$$\begin{aligned}
\Delta^{L+T}(\alpha_s = .15, D_2^{L+T}) &= 0.952 + 0.182\ h + 0.0664\ h^2 + 0.0278\ h^3 + 0.0249\ h^4 \\
&= 0.952,\ 1.134,\ 1.2004,\ 1.2282,\ 1.253 \\
\Delta^{L+T}(\alpha_s = .15, \text{direct}) &= 1.05 + 0.118\ h + 0.0453\ h^2 + 0.0201\ h^3 + 0.0174\ h^4 \\
&= 1.05,\ 1.168,\ 1.2133,\ 1.2334,\ 1.251
\end{aligned}$$

$$\begin{aligned}
\Delta_{30}^{L+T}(\alpha_s = .334, D_2^{L+T}) &= 1.19 + 0.571\ h + 0.48\ h^2 + 0.416\ h^3 + 0.625\ h^4 \\
&= 1.19,\ 1.761,\ 2.241,\ 2.657,\ 3.282 \\
\Delta_{30}^{L+T}(\alpha_s = .334, \text{direct}) &= 1.59 + 0.471\ h + 0.413\ h^2 + 0.339\ h^3 + 0.269\ h^4 \\
&= 1.59,\ 2.061,\ 2.474,\ 2.813,\ 3.082
\end{aligned}$$

where 4-loop terms come from PMS estimations for Π_2^{L+T}

BUT! LIFE IS NOT SO SIMPLE!!!

$$\Delta_{20}^{L+T}(\alpha_s = .334, D_2^{L+T}) \;=\; 1.05 + 0.451\ h + 0.327\ h^2 + 0.223\ h^3 + 0.152\ h^4$$
$$=\; 1.05,\ 1.501,\ 1.828,\ 2.05,\ 2.203$$

$$\Delta_{20}^{L+T}(\alpha_s = .334, \text{direct}) \;=\; 1.35 + 0.347\ h + 0.247\ h^2 + 0.12\ h^3 - 0.223\ h^4$$
$$=\; 1.35,\ 1.697,\ 1.944,\ 2.064,\ 1.841$$

one observes rather siginificant contribution from $\mathcal{O}({\alpha_s}^4)$ term: it looks like both (or one of) 2 PT series begin to behave itlsef wildly at this order! (/Kambor, Maltman (2000)/

Parameter	(2,0)	(3,0)	(4,0)	w. aver
Total	$^{+33.6}_{-44.3}$	$^{+25.0}_{-29.5}$	$^{+21.3}_{-23.0}$	
$m_s(\mathcal{O}(a_s^3)$, exact)	92.5	85.3	78.1	82.5 ± 17
$\mathcal{O}(a_s^3)$	$^{-4.6}_{+5.5}$	$^{-6.0}_{+7.6}$	$^{-6.7}_{+8.0}$	
others	$^{+34}_{-44}$	$^{+25}_{-29}$	$^{+20}_{-22}$	
Total	$^{+33.6}_{-44.3}$	$^{+25.0}_{-29.5}$	$^{+21.6}_{-23.0}$	
$m_s(\mathcal{O}(a_s^4)$, PMS)	89.3	76.8	66.5	73.2 ± 17
$\mathcal{O}(a_s^4)$	$^{-3.0}_{+3.2}$	$^{-6.4}_{+8.6}$	$^{-7.6}_{+11.6}$	
others	$^{+34}_{-44}$	$^{+25}_{-30}$	$^{+20}_{-22}$	
Total	$^{+33.4}_{-44.3}$	$^{+25.6}_{-29.8}$	$^{+23.0}_{-23.4}$	

Table 1: An update of the Table 1 of [1] /Gámiz,Jamin,Pic,Prades,Schwab (2004)/ for m_s extracted from recent exp.data of the OPAL collaboration /G.Abbiendi et al, (2004)/ with subtracted longitudinal contribution accoding to [1]. The contour improvement has been done with the Adler function D_2^{L+T}. "others" $\Rightarrow$ all uncertainties (added in quadrature) of the input parameters different from the $\mathcal{O}(a_s^3)$ (or $\mathcal{O}(a_s^4)$) terms in the perturbative contribution. The last colomun shows a weighted average over the different moments (as the individual error for a given moment we have chosen the larger one)

Parameter	(2,0)	(3,0)	(4,0)	w. aver
$m_s(\mathcal{O}(a_s^3), \text{exact})$	92.4	83.0	74.2	79.6 ± 17
$\mathcal{O}(a_s^3)$	$^{-2.6}_{+2.8}$	$^{-4.6}_{+5.5}$	$^{-5.6}_{+7.3}$	
others	$^{+34}_{-44}$	$^{+24}_{-29}$	$^{+20}_{-22}$	
Total	$^{+33.6}_{-44.3}$	$^{+25.0}_{-29.5}$	$^{+21.3}_{-23.0}$	
$m_s(\mathcal{O}(a_s^4), \text{PMS})$	97.9	79.3	65.7	74.7 ± 17
$\mathcal{O}(a_s^4)$	$^{-5.5}_{+6.5}$	$^{-3.3}_{+3.7}$	$^{-6}_{+8.4}$	
others	$^{+34}_{-44}$	$^{+24}_{-29}$	$^{+20}_{-22}$	
Total	$^{+41}_{-45}$	$^{+24.0}_{-29}$	$^{+22}_{-23}$	

Table 2: The same as Table I but with the contour improvement done directly for Π_2^{L+T}

$\Rightarrow$ at $\mathcal{O}(a_s^4)$ weighted averages of both tables are close and lead to

$$m_s(M_\tau)_{\text{5-loops}} = 74 \ \pm 23 \text{ MeV}$$

which should be compared to

$$m_s(M_\tau)_{\text{4-loops}} = 84 \ \pm \ 23 \text{ MeV} \ \ \text{according to /Gámiz, et al (2004)/}$$

Summary: $\boldsymbol{\alpha_s}$

- α_s^4-terms for $R_{e^+e^-}$ and R_τ are important for improved determination of α_s

- subleading n_f terms are available

- reasonable agreement with previous estimates

  $\Rightarrow$ improved value for α_s

- complete calculation of α_s^4-terms for R_τ and $R_{e^+e^-}$ in the massless limit is possible and is currently under way

- difference between CIPT and FOPT results seems to persist in higher orders

Summary: $\boldsymbol{m_s}$

- analytical QCD result for at $\boldsymbol{m_s^2 \alpha_s{}^3}$ order contribution to R_τ is available

- comparison of the exact result to predictions from various "optimization schemes" demonstrate striking success (relative accuracy around 1(**!**) percent) of PMS and FAC

- the success strongly suggests **to consider and to use** the PMS prediction for $\boldsymbol{m_s^2 \alpha_s{}^4}$ as quite reliable one

- pure convergence of the PT series for $m_s \Rightarrow$ requires new ideas (clever than $L + T$ choice of the integration weight? /Kambor, Maltman (2000)/)

- accurate measurements of lower moments of $R_{\tau,S}$ are important to decrease unphysical dependence of m_s from the moment

- no way to compute $\boldsymbol{m_s^2 \alpha_s{}^4}$-term in any foreseeable future